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Determination of the chemical potential and the energy of the $\nu = 1/2$ FQHE system for low temperatures

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Abstract. We consider the energy density of a spin polarized $\nu = 1/2$ system for low temperatures. We show that due to the elimination of the magnetic field and the field of the positive background charge in the calculation of the grand canonical potential of Chern-Simons systems through a mean field formalism one gets corrections to the well known equations which determine the chemical potential and the energy from the grand canonical potential. We use these corrected equations to calculate the chemical potential and the energy of the $\nu = 1/2$ system at low temperatures in two different approximations.

PACS. 71.10.Pm Fermions in reduced dimensions (anyons, composite fermions, Luttinger liquid, etc.) – 73.43.-f Quantum Hall effects – 71.27.+a Strongly correlated electron systems; heavy fermions

1 Introduction

The combination of an electronic interaction and a strong magnetic field in a two-dimensional electron system yields a rich variety of phases. These are best classified by the filling factor ν, which is the electron density divided by the density of a completely filled Landau level. In this work we mainly consider energy calculations on systems with filling factor $\nu = 1/2$. These systems are most suitably described by the Chern-Simons theory. Within this theory one gets new quasi-particles (composite fermions). In the case of filling fraction $\nu = 1/2$ every electron gets two magnetic flux quantums to build a composite fermion which does not see any magnetic field in first approximation (mean field). A field theoretical language for this scenario was first established by Halperin, Lee, Read (HLR) (1992) [1] as well as Kalmeyer and Zhang (1992) [2] for the $\nu = 1/2$ system. The interpretation of many experiments supports this composite fermion picture. We mention transport experiments with quantum (anti-) dots [3], and focusing experiments [4] here. An overview of further experiments can be found in [5].

HLR studied many physical quantities within the random-phase approximation (RPA). Most prominent among these is the effective mass of the composite fermions which they found to diverge at the Fermi surface [1,6]. Besides the theory of HLR there are other alternative formulations of the composite fermionic picture which are mainly based on a gauge transformation of the Chern-Simons Hamiltonian [7,8].

In the following we will apply the Chern-Simons theory of HLR to calculate the chemical potential μ and the ground state energy U of the spin polarized $\nu = 1/2$ system for low temperatures $T \geq 0$. In [9,10] we calculated the grand canonical potential of the $\nu = 1/2$ system for temperature $T = 0$ in RPA. The energy was calculated from this potential by fixing the chemical potential to the value of a free electron system. Since the RPA consists of no anomalous Feynman graphs this is motivated as a good approximation for $T = 0$ by the Luttinger-Ward theorem [11]. For temperature $T \geq 0$ one has to determine the chemical potential μ by other methods. In the thermodynamic theory the chemical potential μ and the energy U is calculated by the equations $-\partial/(\partial\mu)\,\Omega = \rho$ and $\partial/(\partial\beta)(\beta\Omega) + \mu\rho = U$ from the grand canonical potential Ω. ρ is the density of the electrons. We calculated in [9,10] the grand canonical potential of the Hamiltonian of electrons in a magnetic field B subjected by an electron-electron interaction and an interaction with a positive background field ρ_B as a function of μ, β. We eliminated B and ρ_B from the grand canonical potential Ω by the constraint $B = 2\pi\tilde{\phi}\rho$ and $\rho_B = \rho$. By doing this, it is not clear whether the relations $-\partial/(\partial\mu)\Omega = \rho$ and $\partial/(\partial\beta)(\beta\Omega) + \mu\rho = U$ are further valid for the potential where these two external variables are eliminated through an implicit function. We notice that the elimination of ρ_B by $\rho_B = \rho$ is also a standard method in calculating the grand canonical potential of the Coulomb gas [12]. We will show in Section 2 that the above equations for the determination of μ and U from the grand canonical potential is further valid in the case of the Coulomb gas. In the case of the $\nu = 1/\tilde{\phi}$ Chern-Simons gas we will show that the above relations get additional terms of correction which are not too small. We will show that these additional terms can

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be calculated from the variable eliminated grand canonical potential.

In [9,10] we calculate the e^2-part of the ground state energy where e^2 is the coupling constant of the Coulomb potential $V^{ee} = e^2/r$. This is the first term of an e^2-expansion of the Coulomb energy and agrees with the Coulomb energy calculated for electrons which 'live' in the lowest Landau level. We now have the following two options to expand the calculation to low temperatures $T > 0$.

$$e^2\sqrt{\rho} \ll \frac{1}{\beta} \ll \frac{\rho}{m}, \tag{1}$$

$$\frac{1}{\beta} \ll e^2\sqrt{\rho} \ll \frac{\rho}{m}\,. \tag{2}$$

Here m is the band mass of the electrons and $\beta = 1/(k_\mathrm{B}T)$ where k_B is the Boltzmann constant. The first expansion (1) results in an expansion in $m/(\rho^2\beta)$ for the temperature corrections of every $T = 0$-term. The second expansion (2) results in an expansion in $1/(e^2\sqrt{\rho}\beta)$ for the temperature corrections.

In a perturbational treatment of the Chern-Simons theory the grand canonical potential is given by the first expansion (1) (one calculates the grand canonical potential around a free grand canonical potential Ω_0. The low temperature expansion of Ω_0 correspondence to an expansion which is valid in the parameter range (1)). Thus we will calculate the grand canonical potential at first in the Hartree-Fock approximation. It is well known [14], that the chemical potential (for $\tilde{\phi} = 2$) respectively the ground state energy contains exponentially vanishing temperature corrections for small T for a system of electrons in a magnetic field without Coulomb interaction. One could not obtain this feature by perturbational methods. This behaviour of the temperature corrections of the chemical potential and the energy is no longer valid for electrons in a magnetic field taking into consideration the Coulomb interaction. Therefore, in the following we will calculate only the Coulomb part of the ground state energy. It is an interesting property of the Coulomb energy of the Chern-Simons gas that the Coulomb exchange graph yields most of the energy within the RPA [10]. We will calculate in this paper the Coulomb energy of the Hartree-Fock approximation of the Chern-Simons theory for low temperatures $T > 0$. This graph was calculated by Isihara and Toyoda [13] for the case of the Coulomb gas including the spin degrees of freedom. There are two reasons for a difference between the Coulomb part of the Hartree-Fock energy of the Chern-Simons gas and the Coulomb gas. First, the Chern-Simons gas is spin polarized which is not the case for the Coulomb gas. This results in a much less steeper slope of the Coulomb energy as a function of the temperature for the Chern-Simons gas in comparison to the Coulomb gas. Second, as mentioned above, we have different equations for the determination of the chemical potential and the energy for these two systems. We will show that this difference decreases the slope of the energy curve of the Chern-Simons gas further in comparison to the Coulomb gas. This decreasing of the slope can be explained by perturbational arguments.

It is possible to calculate the effective mass of an interacting system through comparing its specific heat with the specific heat of an electron gas without Coulomb interaction for $B = 0$. By comparing the two expansions (1) and (2) with the scaling of the effective mass [1] $m^* \sim \sqrt{\rho}/e^2$ of the Chern-Simons theory we obtain that only the second expansion leads to the correct scaling. HLR [1] and Kim, Lee [15] calculate the effective mass in the perturbational Chern-Simons theory by a resummation of the grand canonical potential through the RPA. With the help of this grand canonical potential they got a logarithmic diverging effective mass. We will calculate the chemical potential and the energy in the parameter range (2) by using the temperature corrections of the grand canonical potential calculated by HLR and Kim, Lee. We will get the same energy as in the simplification [15] by setting the chemical potential on the lowest Landau level and neglecting the additional terms of correction in determining the energy from the grand canonical potential.

The paper is organized as follows: In Section 2 we calculate the equations determining the chemical potential and the ground state energy from a grand canonical potential of the $\nu = 1/\tilde{\phi}$ system where the fields B and ρ_B are eliminated. In Section 3 we calculate the Coulomb energy and the chemical potential of the $\nu = 1/2$ system for temperatures $T > 0$ within the two parameter ranges (1) and (2).

2 The determination of the chemical potential and the energy in $\nu = 1/\tilde{\phi}$ Chern-Simons systems

In this section we show that one gets new equations to determine the chemical potential and the energy from the grand canonical potential in Chern-Simons theories where the external fields, *i.e.* the magnetic field B and the density of the positive background ρ_B are eliminated by some mean field conditions. We will show further that this is not the case for the Coulomb gas. In the following we will consider interacting spin polarized electrons moving in two dimensions in a strong magnetic field B directed in the positive z-direction of the system. The electronic density of the system is chosen such that the lowest Landau level of a non-interacting system is filled to a fraction $\nu = 1/\tilde{\phi}$ where $\tilde{\phi}$ is an even number. We are mainly interested in $\tilde{\phi} = 2$. The Hamiltonian of electrons in a magnetic field is given by

$$H(\mathbf{A}, e^2) = \int \mathrm{d}^2 r \left[\frac{1}{2m} \big| \big(-\mathrm{i}\boldsymbol{\nabla} + \mathbf{A} \big) \Psi(\mathbf{r}) \big|^2 + \frac{1}{2} \int \mathrm{d}^2 r' \Big\{ : (|\Psi(\mathbf{r})|^2 - \rho_B) V^{ee}(|\mathbf{r} - \mathbf{r}'|)(|\Psi(\mathbf{r}')|^2 - \rho_B) : \Big\} \right]. \tag{3}$$

Here $\Psi^+(\mathbf{r})$ creates (and $\Psi(\mathbf{r})$ annihilates) an electron with coordinate $\mathbf{r}$. $: O :$ is the normal ordering of the operator O. $V^{ee}(r) = e^2/r$ is the Coulomb interaction where

$e^2 = q_e^2/\epsilon$. q_e is the charge of the electrons and ϵ is the dielectric constant of the background field ρ_B. $\mathbf{A}(\mathbf{r})$ is the vector potential $\mathbf{A} = 1/2\,\mathbf{B} \times \mathbf{r}$ and $\mathbf{B}$ is a homogeneous magnetic field in z-direction $\mathbf{B} = B\mathbf{e}_z$ where $\mathbf{e}_z$ is the unit vector in z-direction. We suppose throughout this paper that B is a positive number. We used the convention $\hbar = 1$ and $c = 1$ in the above formula (3). Furthermore, we set $q_e = 1$ for the coupling of the magnetic potential to the electrons. After performing a Chern-Simons transformation [16] of the electronic wave function one gets the Hamiltonian of the composite fermions as:

$$H_{\mathrm{CS}}(\mathbf{A}, e^2) = \int \mathrm{d}^2 r \left[\frac{1}{2m} \big| \big(-\mathrm{i}\boldsymbol{\nabla} + \mathbf{A} + \mathbf{a}_{\mathrm{CS}} \big) \Psi(\mathbf{r}) \big|^2 + \frac{1}{2} \int \mathrm{d}^2 r' \Big\{ \!: (|\Psi(\mathbf{r})|^2 - \rho_B) V^{ee}(|\mathbf{r} - \mathbf{r}'|)(|\Psi(\mathbf{r}')|^2 - \rho_B) :\! \Big\} \right]. \tag{4}$$

The Chern-Simons operator $\mathbf{a}_{\mathrm{CS}}$ is defined by $\mathbf{a}_{\mathrm{CS}}(\mathbf{r}) = \tilde{\phi} \int \mathrm{d}^2 r' \mathbf{f}(\mathbf{r} - \mathbf{r}')\Psi^+(\mathbf{r}')\Psi(\mathbf{r}')$. Here $\Psi^+(\mathbf{r})$ creates (and $\Psi(\mathbf{r})$ annihilates) a composite fermion with coordinate $\mathbf{r}$. The function $\mathbf{f}(\mathbf{r})$ is given by $\mathbf{f}(\mathbf{r}) = -\mathbf{e}_z \times \mathbf{r}/r^2$. From this Hamiltonian we obtain in the case of filling fraction $\nu = 1/\tilde{\phi}$ and electro neutrality, *i.e.* $B = 2\pi\tilde{\phi}\langle\hat{\rho}\rangle$ and $\rho_B = \langle\hat{\rho}\rangle$, that in the perturbation theory of H_{CS} the Hartree couplings $\mathbf{A} + \tilde{\phi} \int \mathrm{d}^2 r'\, \mathbf{f}(\mathbf{r} - \mathbf{r}')\langle\Psi^+(\mathbf{r}')\Psi(\mathbf{r}')\rangle$ and $\int \mathrm{d}^2 r'\, V^{ee}(|\mathbf{r} - \mathbf{r}'|)(\langle\Psi^+(\mathbf{r}')\Psi(\mathbf{r}')\rangle - \rho_B)$ are zero. If we calculate the Feynman graphs of the theory under these conditions we eliminate the external fields B and ρ_B of the partition function $Z = \mathrm{Tr}[\mathrm{e}^{-\beta(H_{\mathrm{CS}}(\mathbf{A}, e^2) - \mu\hat{N})}] = \mathrm{Tr}[\mathrm{e}^{-\beta(H(\mathbf{A}, e^2) - \mu\hat{N}}]$. Here $\mathrm{Tr}[\cdot]$ is the trace of the argument. $\hat{N}$ is the particle number operator $\hat{N} = \int \mathrm{d}^2 r\, \Psi^+(\mathbf{r})\Psi(\mathbf{r})$. The variables left in Ω are μ and β. In the following we will deal with three different systems with a growing degree of complication.

At first we deal with the Coulomb gas ($\mathbf{A} = 0$ in (3)). In perturbational calculations of this system one often calculates the Feynman graphs under the condition $\rho_B = \langle\hat{\rho}\rangle$ [12]. $\hat{\rho}(\mathbf{r})$ is the density operator $\Psi^+(\mathbf{r})\Psi(\mathbf{r})$. $\langle\cdot\rangle$ is the expectation value of the Gibb's potential of the Hamiltonian. For a homogeneous system we have $\langle\hat{\rho}(\mathbf{r})\rangle = \langle\Psi^+(\mathbf{r})\Psi(\mathbf{r})\rangle = \langle\hat{\rho}\rangle$. Thus one calculates a partition function $Z'(\mu, \beta)$ from $Z(\mu, \beta, \rho_B) = \mathrm{Tr}[\mathrm{e}^{-\beta(H(0, e^2) - \mu\hat{N})}]$ by

$$Z'(\mu, \beta) := Z\left(\mu, \beta, \rho_B(\mu, \beta)\right), \tag{5}$$

$$\rho_B(\mu, \beta) := \frac{1}{A\beta} \frac{\partial}{\partial\mu} \log\left(Z\right)\left(\mu, \beta, \rho_B(\mu, \beta)\right). \tag{6}$$

A is the area of the system. We employed in (5) and (6) the mathematical notation for the ordering of the derivation and insertion of the arguments of the functions. This means for example for equation (6) that we have to partially derivate at first the function Z depending on the variables (μ, β, ρ_B). Afterwards we have to insert the expressions given in the function brackets. We have to point out explicitly that $\rho_B(\mu, \beta)$ is a function of (μ, β) which is defined by equation (6). We now want to determine the chemical potential μ and the energy U from Z'. At first we deal with the derivation of Z with respect to μ:

$$\begin{aligned} \frac{1}{\beta} \frac{\partial}{\partial\mu} \log\left(Z\right)\left(\mu, \beta, \rho_B(\mu, \beta)\right) &= \frac{1}{\beta} \frac{\partial}{\partial\mu} \log\left(Z'\right)\left(\mu, \beta\right) + K_1 \\ &= A\left\langle\hat{\rho}\right\rangle. \end{aligned} \tag{7}$$

K_1 is defined by

$$\begin{aligned} K_1 = -\int_A \int_A \mathrm{d}^2 r \mathrm{d}^2 r' (\langle\hat{\rho}(\mathbf{r})\rangle - \rho_B) V^{ee}(\boldsymbol{r} - \mathbf{r}') \\ \times \frac{\partial}{\partial\mu} \rho_B(\mu, \beta) \bigg|_{\langle\hat{\rho}\rangle = \rho_B} = 0. \end{aligned} \tag{8}$$

Thus we get the following equations determining the chemical potential μ and the energy U of the Coulomb system $-\frac{\partial}{\partial\mu}\Omega'(\mu, \beta) = \langle\hat{\rho}\rangle$ and $U = \frac{\partial}{\partial\beta}(\beta\Omega')(\mu, \beta) + \mu\langle\hat{\rho}\rangle$. The grand canonical potential $\Omega'(\mu, \beta)$ is defined by $\Omega' = -1/(\beta A)\log(Z')$. From these equations we obtain that in the case of the Coulomb gas we get no correction to the well known equations for the determination of the chemical potential and the energy.

In the following we will derive equations to determine μ and U for the $\nu = 1/\tilde{\phi}$ Chern-Simons gas with no Coulomb interaction and $T = 0$. As mentioned earlier the partition function $Z(\mu, \beta, B) = \mathrm{Tr}[\mathrm{e}^{-\beta(H(\mathbf{A}, 0) - \mu\hat{N})}]$ of this system is usually calculated under the constraint $\langle\hat{\rho}\rangle = B/(2\pi\tilde{\phi})$ giving $Z'(\mu, \beta)$. So Z' is defined by

$$Z'(\mu, \beta) := Z\left(\mu, \beta, B(\mu, \beta)\right), \tag{9}$$

$$B(\mu, \beta) := (2\pi\tilde{\phi}) \frac{1}{A\beta} \frac{\partial}{\partial\mu} \log\left(Z\right)\left(\mu, \beta, B(\mu, \beta)\right). \tag{10}$$

From these definitions we get for the derivate of the partition function Z with respect to μ

$$\begin{aligned} \frac{1}{\beta} \frac{\partial}{\partial\mu} \log(Z)(\mu, \beta, B(\mu, \beta)) &= \frac{1}{\beta} \frac{\partial}{\partial\mu} \log(Z')(\mu, \beta) + K_2 \\ &= A\langle\hat{\rho}\rangle. \end{aligned} \tag{11}$$

K_2 is defined by

$$K_2 = -\frac{1}{2\pi} \int_A \int_A \mathrm{d}^2 r\, \mathrm{d}^2 r'\, \langle\hat{\mathbf{j}}\rangle(\mathbf{r})\mathbf{f}(\mathbf{r} - \mathbf{r}') \frac{\partial}{\partial\mu} B(\mu, \beta). \tag{12}$$

$\hat{\mathbf{j}}(\mathbf{r})$ is the second quantized current operator $(1/2m)(\Psi^+(\mathbf{r})(-\mathrm{i}\boldsymbol{\nabla} + \mathbf{A}(\mathbf{r}))\Psi(\mathbf{r}) + [(-\mathrm{i}\boldsymbol{\nabla} + \mathbf{A}(\mathbf{r}))\Psi(\mathbf{r})]^+\Psi(\mathbf{r}))$. It is clear that the expectation value of the current operator in (12) is built with respect to the Gibb's operator of the Hamiltonian $H(\mathbf{A}, 0)$ (3). At first glance one may think that K_2 is zero because we have $\langle\hat{\mathbf{j}}\rangle(\mathbf{r}) = 0$ for $A \to \infty$. We have to be careful with such argumentation because the multiplicative term $\mathbf{f}(\mathbf{r} - \mathbf{r}')$ in (12) is not integrable for $|\mathbf{r} - \mathbf{r}'| \to \infty$. The right way to calculate K_2 is to evaluate $\langle\hat{\mathbf{j}}\rangle(\mathbf{r})$ for a finite system. Then after integrating (12) for this finite system one should take the limit $A \to \infty$. For a finite system we have nothing but a current at the edge of the system. This

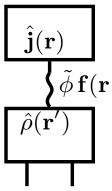

Fig. 1. Density graphs, which contribute to the μ-correction.

ring current was calculated perturbationally by Shankar and Murthy [7]. With the help of this ring current one could calculate K_2. In the following we will calculate K_2 by another method which results in the same value for K_2 but which does not use any perturbational results. Using the one particle eigen functions u_{0l} of an electron in a magnetic field in the lowest Landau level (symmetric gauge) we get

$$-\frac{1}{2\pi}\int \mathrm{d}^2r\,\mathrm{d}^2r'\left\langle u_{0k}|\hat{\mathbf{j}}(\mathbf{r})|u_{0l}\right\rangle \mathbf{f}(\mathbf{r}-\mathbf{r}') = \frac{1}{2m}\delta_{k,l}. \quad (13)$$

In (13) we supposed that u_{0k} is orthogonal to u_{0l} for $k \neq l$. For deriving (13) we used $\mathbf{A}(\mathbf{r}) = -B/(2\pi)\int \mathrm{d}^2r'\mathbf{f}(\mathbf{r}-\mathbf{r}') = B/2(-y,x)$.

By using (10, 11, 12) and (13) we get

$$-\frac{\partial}{\partial\mu}\Omega'(\mu,\beta) + \frac{\pi\tilde{\phi}}{m}\langle\hat{\rho}\rangle\frac{\partial}{\partial\mu}\langle\hat{\rho}\rangle(\mu,\beta) = \langle\hat{\rho}\rangle. \quad (14)$$

Here we used $B(\mu,\beta) = 2\pi\tilde{\phi}\langle\hat{\rho}\rangle(\mu,\beta)$. So we have to solve a differential equation to get the chemical potential for a given density $\langle\hat{\rho}\rangle$. For the determination of the energy U from Ω' we get with the help of a similar derivation as above

$$U = \frac{\partial}{\partial\beta}(\beta\Omega')(\mu,\beta) - \frac{\pi\tilde{\phi}}{m}\beta\langle\hat{\rho}\rangle\frac{\partial}{\partial\beta}\langle\hat{\rho}\rangle(\mu,\beta) + \mu\langle\hat{\rho}\rangle. \quad (15)$$

Thus we obtain from the equations (14) and (15) an additional term in comparison to the equations which determine the energy and the chemical potential from the grand canonical potential $\Omega = -1/(\beta A)\log(Z)$. In Figure 1 we show density graphs of the Hamiltonian $H_{\mathrm{CS}}(\mathbf{A},e^2)$ which contributes to this additional term. Since $\langle\hat{\rho}\rangle$ is finite for $T = 0$ the second term in (15) is zero. It is well known that the energy and the chemical potential of electrons in a magnetic field at $T = 0$ are given by $U = \mu\rho$ and $\mu = \pi\rho\tilde{\phi}/m$. So we get from (14) and (15)

$$-\frac{\partial}{\partial\mu}\Omega'(\mu,\beta) = 0, \quad \frac{\partial}{\partial\beta}(\beta\Omega')(\mu,\beta) = 0. \quad (16)$$

We obtain from these equations that in the case of an interaction free Chern-Simons system the correction to the equations determining μ and U are not neglectable. This should be also the case for a Chern-Simons system taking into account the Coulomb interaction.

In the following part of this section we will derive the equations for getting μ and U of the $\nu = 1/\tilde{\phi}$ Chern-Simons gas for low temperatures taking into consideration the Coulomb interaction. In the mean field treatment of the $\nu = 1/\tilde{\phi}$ Chern-Simons theory one usually calculates from the partition function $Z(\mu,\beta,\rho_B,B) = \mathrm{Tr}[\mathrm{e}^{-\beta(H(\mathbf{A},e^2)-\mu\hat{N})}]$ the following reduced partition function

$$Z'(\mu,\beta) = Z\left(\mu,\beta,\rho_B(\mu,\beta),B(\mu,\beta)\right). \quad (17)$$

$\rho_B(\mu,\beta)$, $B(\mu,\beta)$ are defined by (6, 10). It is now possible to make the same derivations for this system as in the case of the Chern-Simons system with no Coulomb interaction. By doing this we get equation (11). Unfortunately it is then hard to calculate the expectation value (12) for this system because we have a mixing of higher Landau levels. Nevertheless one can get the equations which determine μ and U by scaling arguments. This will be done in the following. From the definition of Z' we get

$$\begin{aligned}
&\frac{1}{A\beta}\frac{\partial}{\partial\mu}\log\left(Z\right)\left(\mu,\beta,\rho_B(\mu,\beta),B(\mu,\beta)\right) = \\
&\frac{1}{A\beta}\Bigg[-\frac{\partial}{\partial B}\log\left(Z\right)\left(\mu,\beta,\rho_B(\mu,\beta),B(\mu,\beta)\right)\frac{\partial}{\partial\mu}B(\mu,\beta) \\
&\qquad -\frac{\partial}{\partial\rho_B}\log\left(Z\right)\left(\mu,\beta,\rho_B,B(\mu,\beta)\right)\frac{\partial}{\partial\mu}\rho_B(\mu,\beta) \\
&\qquad\qquad +\frac{\partial}{\partial\mu}\log\left(Z'\right)(\mu,\beta)\Bigg]. \quad (18)
\end{aligned}$$

As in the case of the Coulomb gas the second term on the right hand side is zero. Thus we have to calculate the first term on the right hand side of equation (18). With the help of the definitions

$$\begin{aligned}
a_{np} &= \int \mathrm{d}^2r\; u_{np}(\mathbf{r})\Psi(\mathbf{r}), \\
V^{ee}_{n_1p_1,n_2p_2,n_3p_3,n_4p_4} &= \int \mathrm{d}^2r\mathrm{d}^2r'\; V^{ee}(\mathbf{r}-\mathbf{r}') \\
&\quad\times u^*_{n_1p_1}(\mathbf{r})u_{n_2p_2}(\mathbf{r})u^*_{n_3p_3}(\mathbf{r}')u_{n_4p_4}(\mathbf{r}') \quad (19)
\end{aligned}$$

the operators $H_0(B)$ and $H_{ee}(B)$ are defined by

$$H_0(B) = \sum_{n,p}\frac{B}{m}\left(n+\frac{1}{2}\right)a^+_{np}a_{np}, \quad (20)$$

$$\begin{aligned}
H_{ee}(B) &= \frac{1}{2}\sum_{\substack{n_1p_1,n_2p_2,\\ n_3p_3,n_4p_4}} V^{ee}_{n_1p_1,n_2p_2,n_3p_3,n_4p_4} \quad (21)\\
&\quad\times :\left(a^+_{n_1p_1}a_{n_2p_2}-\rho_B\right)\left(a^+_{n_3p_3}a_{n_4p_4}-\rho_B\right): .
\end{aligned}$$

u_{np} are the one particle wave functions in the symmetric gauge for the magnetic field B in the nth landau level [17]. With the help of these operators we get for the grand canonical potential $\Omega = -1/(\beta A)\log(Z)$

$$\begin{aligned}
\Omega(\mu,\beta,\rho_B,B) &= -\frac{1}{\beta}\lim_{A\to\infty}\frac{1}{A} \\
&\quad\times\log\left[\mathrm{Tr}\left[\mathrm{e}^{-\beta(H_0(B)+H_{ee}(B)-\mu\hat{N})}\right]_{B,A}\right]. \quad (22)
\end{aligned}$$

$\mathrm{Tr}[...]_{B,A}$ is the trace of slater determinants of Landau functions which have their support in the area A. The number of Landau functions in this area is proportional to B [17]. With the help of a length scaling $L \rightarrow L/\sqrt{B}$ (*i.e.* we implement the trace in (22) in the Landau basis and make the substitution $\mathbf{r} \rightarrow \mathbf{r}/\sqrt{B}$) we get

$$\Omega(\mu, \beta, \rho_B, B) = -\frac{B}{\beta} \lim_{A\rightarrow\infty} \frac{1}{A} \times \log\left[\mathrm{Tr}\left[\mathrm{e}^{-\beta(BH_0(\frac{B}{|B|})+\sqrt{B}H_{ee}(\frac{B}{|B|})-\mu\hat{N})}\right]_{\frac{B}{|B|},A}\right]. \quad (23)$$

Thus we get

$$\frac{\partial}{\partial B}\Omega(\mu, \beta, \rho_B, B) = \frac{1}{B}\left(\langle H_0(B)\rangle + \frac{1}{2}\langle H_{ee}(B)\rangle + \Omega(\mu, \beta, \rho_B, B)\right). \quad (24)$$

With the help of (18) and $\langle\hat{\rho}\rangle = 1/(2\pi\phi)B(\mu, \beta)$ we get for the equation determining the chemical potential μ from Ω'

$$\left(\langle H_0(B)\rangle + \frac{1}{2}\langle H_{ee}(B)\rangle + \Omega'(\mu, \beta)\right)\frac{1}{\langle\hat{\rho}\rangle}\frac{\partial}{\partial\mu}\langle\hat{\rho}\rangle(\mu, \beta) - \frac{\partial}{\partial\mu}\Omega'(\mu, \beta) = \langle\hat{\rho}\rangle. \quad (25)$$

Similar as above we get for the energy of the $\nu = 1/\tilde{\phi}$ system

$$U = -\left(\langle H_0(B)\rangle + \frac{1}{2}\langle H_{ee}(B)\rangle + \Omega'(\mu, \beta)\right)\beta\frac{1}{\langle\hat{\rho}\rangle}\frac{\partial}{\partial\beta}\langle\hat{\rho}\rangle(\mu, \beta) + \frac{\partial}{\partial\beta}(\beta\Omega')(\mu, \beta) + \mu\langle\hat{\rho}\rangle. \quad (26)$$

In the case of a determination of the energy and the chemical potential to order e^2 it is possible to get $\langle H_0\rangle + \frac{1}{2}\langle H_{ee}\rangle$ from Ω' for low temperatures. In the following we will derive this relation at first for $T = 0$. Afterwards we will generalize the results to the case of low temperatures. It is well known from perturbation theory that the ground state wave function has only higher Landau level components scaling with e^2. So we get for temperature $T = 0$

$$\langle H_0(B)\rangle = \pi\tilde{\phi}\frac{\langle\hat{\rho}\rangle^2}{m} + O(e^4). \quad (27)$$

For $T > 0$ it is no longer correct that the second term of the e^2-expansion of $\langle H_0(B)\rangle$ scales like e^4. Nevertheless because of the discreteness of the Landau levels we obtain that these T-corrections are exponentially vanishing for $T \rightarrow 0$. The reason for this is that due to the expansions (1) and (2) there is a coupling factor e_0^2 such that we get no overlap between the energy eigen values of wave functions resulting from the lowest Landau level by Coulomb perturbations and wave functions resulting from higher Landau levels for all $e^2 \leq e_0^2$. Since $1/\beta \ll \rho/m$ for both expansions (1) and (2) we get that the kinetic energy resulting from wave functions of higher Landau levels by Coulomb perturbations are suppressed exponentially with $\exp[-\beta\rho/m]$. Since we are only interested in polynomial T-contributions to the energy we can neglect these T-corrections in (27). Under the consideration of equation (27) we can get $\langle H_{ee}\rangle$ up to order e^2 from Ω by

$$[\langle H_{ee}(B)\rangle]_{e^2} = [U(\langle\hat{\rho}\rangle, \beta)]_{e^2} = \left[\frac{\partial(\beta\Omega)}{\partial\beta}(\mu(\langle\hat{\rho}\rangle, \beta), \beta, \langle\hat{\rho}\rangle, 2\pi\tilde{\phi}\langle\hat{\rho}\rangle) + \mu(\langle\hat{\rho}\rangle, \beta)\langle\hat{\rho}\rangle\right]_{e^2}. \quad (28)$$

We denote by $[...]_{e^2}$ the e^2-part of the bracket for $T = 0$. In the case of low temperatures T we have to distinguish between the two low temperature expansions (1) and (2). In the case of the temperature expansion (1) $[...]_{e^2}$ is given by the $e^2(\rho)^{3/2}/(\beta\rho/m)^n$ $(n \geq 0)$ terms of the bracket. For the temperature expansion (2) $[...]_{e^2}$ is given by the $e^2(\rho)^{3/2}/(\beta e^2\sqrt{\rho})^n$ $(n \geq 0)$ terms of the bracket. We may now determine the chemical potential and the energy in every order of $1/\beta$ by using successive the equations (25–28) under consideration that $\lim_{\beta\rightarrow\infty}\frac{\partial}{\partial\beta}\langle\hat{\rho}\rangle = O(1/\beta^{1+\epsilon})$ for a number ϵ with $\epsilon > 0$. Thus we can calculate the chemical potential and the energy to every order in $1/\beta$ through a finite number of successive insertions of these equations. Doing this for $T = 0$ we get

$$\left(\frac{1}{2\langle\hat{\rho}\rangle}\left[\frac{\partial}{\partial\beta}(\beta\Omega')(\mu(\langle\hat{\rho}\rangle, \infty), \infty) + \mu(\langle\hat{\rho}\rangle, \infty)\langle\hat{\rho}\rangle\right]_{e^2} + \frac{1}{\langle\hat{\rho}\rangle}\Omega'(\mu, \infty) + (\pi\tilde{\phi})\frac{\langle\hat{\rho}\rangle}{m}\right)\frac{\partial}{\partial\mu}\langle\hat{\rho}\rangle(\mu, \beta) - \frac{\partial}{\partial\mu}\Omega'(\mu, \infty) = \langle\hat{\rho}\rangle \quad (29)$$

and

$$U = \frac{\partial(\beta\Omega')}{\partial\beta}(\mu, \infty) + \mu\langle\hat{\rho}\rangle. \quad (30)$$

By reducing equation (29) to the Chern-Simons gas without Coulomb interaction we get a difference of a summand $\frac{1}{\langle\hat{\rho}\rangle}\Omega'(\mu, \infty)\frac{\partial}{\partial\mu}\langle\hat{\rho}\rangle(\mu, \infty)$ between the equations (14) and (29). The grand canonical potential for an electron gas in a magnetic field without Coulomb interaction is given by

$$\Omega_B(\mu, \beta, B) = -\frac{1}{2\pi}\frac{B}{\beta} \times \sum_n \log\left(1 + \exp\left[-\beta\left(\left(n + \frac{1}{2}\right)\frac{B}{m} - \mu\right)\right]\right). \quad (31)$$

The chemical potential at low temperatures for the $\nu = 1/2$ gas is given by [14] $\mu = \frac{(2\pi\rho)}{m} + O(\mathrm{e}^{-\beta\rho/m})$. Since $\lim_{\beta\rightarrow\infty}\Omega'(\mu(\rho, \beta), \beta) = O(1/\beta)$ we obtain for $T = 0$ that the equations (14) and (29) are in accordance.

3 The chemical potential and the energy of the $\nu = 1/2$ Chern-Simons system for T ≥ 0

In this section we will calculate the chemical potential and the energy of the $\nu = 1/2$ system for low temperatures T and $e^2\sqrt{\rho} \ll \rho/m$. As written in the introduction there are two parameter ranges for an expansion of the energy and the chemical potential. These are given in (1) and (2). In the following we will calculate in subsection A the chemical potential and the energy in the parameter range (1). In subsection B we will calculate the chemical potential and the energy within the parameter range (2).

3.1 The chemical potential and the energy of the $\nu = 1/2$ Chern-Simons system for $e^2\sqrt{\rho} \ll 1/\beta \ll \rho/m$

As mentioned in the introduction the Chern-Simons perturbations theory has its validity in the parameter range (1). Thus it is possible to calculate the energy and the chemical potential in this parameter range within the Hartree-Fock approximation. As mentioned in the last section one gets exponentially vanishing temperature corrections to the chemical potential and the energy at low temperatures for the $\nu = 1/2$ system without Coulomb interaction. They are not calculable by perturbational methods. Because of this we will calculate only the Coulomb part of the grand canonical potential perturbationally. The exact magnetic part of the grand canonical potential is given by $\Omega_B(\mu, \beta, (2\pi\tilde{\phi})\rho^*)$ (31). In this expression ρ^* is defined by $\rho^* := \partial/(\partial\mu)\Omega_B(\mu, \beta, (2\pi\tilde{\phi})\rho^*)$. As we mentioned above in the case of the $\nu = 1/2$ system we have for low temperatures [14] $\rho^* = m\mu/(2\pi) + O(\mathrm{e}^{-\beta\mu})$. The grand canonical potential is then given by (the exact form of Ω in the Hartree-Fock approximation will be derived later)

$$\begin{aligned}\Omega(\mu, \beta) = \Omega_B(\mu, \beta, (2\pi\tilde{\phi}\rho^*)) + a_1\, e^2 m^{\frac{3}{2}}\mu^{\frac{3}{2}} + a_2\, e^2 m^{\frac{3}{2}}\mu^{\frac{1}{2}}\frac{1}{\beta} \\ + a_3\, e^2 m^{\frac{3}{2}}\frac{1}{\mu^{\frac{1}{2}}}\frac{1}{\beta^2} + a_4\, e^2 m^{\frac{3}{2}}\frac{1}{\mu^{\frac{1}{2}}}\frac{1}{\beta^2}\log\left(\frac{1}{\mu\beta}\right) \\ + O(e^2/(\mu^{3/2}\beta^3)) + O(e^4 m^2\mu^{\frac{1}{2}}).\end{aligned} \tag{32}$$

In the following we will denote $\Omega(\mu, \beta) - \Omega_B(\mu, \beta, (2\pi\tilde{\phi}\rho^*))$ by $\Omega_\mathrm{c}(\mu, \beta)$. We now make the following ansatz for the terms of μ which scales polynomial with $1/\beta$:

$$\begin{aligned}\mu(\rho, \beta) = \pi\tilde{\phi}\frac{\rho}{m} + b_1\, e^2\rho^{\frac{1}{2}} + b_2\, e^2\frac{m}{\rho^{\frac{1}{2}}}\frac{1}{\beta} \\ + b_3 e^2\frac{m^2}{\rho^{\frac{3}{2}}}\frac{1}{\beta^2} + b_4 e^2\frac{m^2}{\rho^{\frac{3}{2}}}\frac{1}{\beta^2}\log\left(\frac{m}{\rho\beta}\right).\end{aligned} \tag{33}$$

In the following we will denote the first term in the sum of $\mu(\rho, \beta)$ by $\mu_0\rho, \beta)$. $\mu(\rho, \beta) - \mu_0(\rho, \beta)$ will be denoted by $\mu_\mathrm{c}(\rho, \beta)$. We get for μ_c inserting (27, 28, 32, 33) in (25) and taking into consideration $\rho^* = m\mu/(2\pi) + O(\mathrm{e}^{-\beta\mu})$:

$$\begin{aligned}-\frac{\partial\mu_0}{\partial\rho}\frac{\partial\Omega_\mathrm{c}}{\partial\mu} + \left(\frac{1}{2}\left(\frac{\partial(\beta\Omega_\mathrm{c})}{\partial\beta} + \mu_\mathrm{c}\rho\right) + \Omega_\mathrm{c}\right)\frac{1}{\rho} \\ + \left(\frac{\mu_\mathrm{c}}{\rho} - \frac{\partial\mu_\mathrm{c}}{\partial\rho}\right)\frac{\Omega_B}{\mu} = \rho\frac{\partial\mu_\mathrm{c}}{\partial\rho\cdot}\end{aligned} \tag{34}$$

We will solve this equation by comparing the coefficients of equal powers of β. If one inspects the equation of the coefficient of $(\beta)^0$ in (34) one notices that the coefficient b_1 vanishes in this equation and one gets a trivial identity which is fulfilled for every chosen b_1. The reason for this indeterminacy of b_1 is given by the partition of Ω in a sum of a pure Chern-Simons term and e^2-corrections to Ω (this could be seen by using (16, 29)). Since this partition is not correct we have no equation to determine the coefficient b_1. For this reason we have to start properly with a Chern-Simons perturbation theory which shows a more complicated mean field. This will be done in a later publication. We now set $b_1 = 0$. This could be justified by the Luttinger-Ward theorem [11] as a good approximation calculating the $T = 0$ energy Ω in RPA because the RPA does not contain any anomalous graphs [9,10]. By comparing the coefficients of the higher order powers of $1/\beta$ in (34) we get for the $\nu = 1/2$ system

$$\begin{aligned}\mu = 2\pi\frac{\rho}{m} - a_2 e^2\frac{\sqrt{2\pi}}{2}\frac{m}{\rho^{\frac{1}{2}}}\frac{1}{\beta} - \left(\frac{a_3}{2} + \frac{3a_2\log(2)}{4}\right) \\ \times e^2\frac{1}{\sqrt{2\pi}}\frac{m^2}{\rho^{\frac{3}{2}}}\frac{1}{\beta^2} - a_4 e^2\frac{1}{2\sqrt{2\pi}}\frac{m^2}{\rho^{\frac{3}{2}}}\frac{1}{\beta^2}\log\left(\frac{m}{2\pi\rho\beta}\right).\end{aligned} \tag{35}$$

With the help of the equations (26, 27) and (28) the energy density up to order e^2 is given by

$$\begin{aligned}U = 2\pi\frac{\rho^2}{m} + \frac{\partial(\beta\Omega_\mathrm{c})}{\partial\beta} - \beta 2\pi\frac{\rho}{m}\left.\frac{\partial\rho_\mathrm{c}(\mu, \beta)}{\partial\beta}\right|_{\mu=\mu(\rho,\beta)} \\ - \beta\frac{\Omega_B}{\rho}\left.\frac{\partial\rho_\mathrm{c}(\mu, \beta)}{\partial\beta}\right|_{\mu=\mu(\rho,\beta)} + \mu_\mathrm{c}\rho.\end{aligned} \tag{36}$$

$\rho_\mathrm{c}(\mu, \beta)$ in (36) is defined by $\rho(\mu, \beta) - m\mu/(2\pi)$. With the help of (32, 36) we get for the energy of the $\nu = 1/2$ system

$$\begin{aligned}U = (2\pi)\frac{\rho^2}{m} + a_1 e^2(2\pi\rho)^{\frac{3}{2}} - \left(\frac{(a_3 + a_4)}{2} + \frac{a_2\log(2)}{4}\right) \\ \times e^2\frac{m^2}{(2\pi\rho)^{\frac{1}{2}}}\frac{1}{\beta^2} - a_4 e^2\frac{m^2}{2(2\pi\rho)^{\frac{1}{2}}}\log\left(\frac{m}{2\pi\rho\beta}\right)\frac{1}{\beta^2}\cdot\end{aligned} \tag{37}$$

In the following we will calculate the Coulomb part of the Hartree-Fock approximation of H_CS (4). This term is equal to the Coulomb exchange Feynman graph given by

$$\mathrm{Ex}^{ee} = \frac{1}{2\,(2\pi)^3}\int \mathrm{d}^2k\mathrm{d}^2k'\, n_\mathrm{F}(k)\, n_\mathrm{F}(k')\, V^{ee}(|\mathbf{k} - \mathbf{k}'|). \tag{38}$$

$n_\mathrm{F}(k)$ is the Fermi factor $n_\mathrm{F}(k) = 1/(1 + \mathrm{e}^{\beta(k^2/(2m)-\mu)})$. Ex^{ee} is also the e^2-part of the grand canonical potential of the (two dimensional) Coulomb gas. This term was calculated by Isihara and Toyoda [13] within the calculation of the grand canonical potential of the two dimensional electron gas. Up to order $1/\beta^2$ it is given by

$$\begin{aligned} \mathrm{Ex}^{ee} \approx & -0.095\, e^2 m^{\frac{3}{2}} \mu^{\frac{3}{2}} \\ & + \left(0.035 - 0.0588 \log\left(\frac{1}{\mu\beta}\right)\right) e^2 \frac{m^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} \frac{1}{\beta^2} \cdot \end{aligned} \tag{39}$$

By using (32, 35, 37) and (39) we get for the chemical potential μ^{HF} and the energy U^{HF} in the Hartree-Fock approximation

$$\begin{aligned} \mu^{\mathrm{HF}} &= 2\pi \frac{\rho}{m} - 0.017\, e^2 \frac{1}{\sqrt{2\pi}} \frac{m^2}{\rho^{\frac{3}{2}}} \frac{1}{\beta^2} \\ & \quad + 0.029\, e^2 \frac{1}{\sqrt{2\pi}} \frac{m^2}{\rho^{\frac{3}{2}}} \frac{1}{\beta^2} \log\left(\frac{m}{2\pi\rho\beta}\right), \qquad (40) \\ U^{\mathrm{HF}} &= (2\pi)\frac{\rho^2}{m} - 0.095\, e^2 (2\pi\rho)^{\frac{3}{2}} + 0.012\, e^2 \frac{m^2}{(2\pi\rho)^{\frac{1}{2}}} \frac{1}{\beta^2} \\ & \quad + 0.029 e^2 \frac{m^2}{(2\pi\rho)^{\frac{1}{2}}} \frac{1}{\beta^2} \log\left(\frac{m}{2\pi\rho\beta}\right). \qquad (41) \end{aligned}$$

For $T = 0$ the Coulomb part of U^{HF} is in good agreement with the Coulomb energy of numerical simulation of electrons on a sphere by Morf and d'Ambrumenil [18] and by Girlich [19] ($\approx -0.1\, e^2(2\pi\rho)^{3/2}$). We can now compare the e^2-part of the energy of the $\nu = 1/2$ system with the e^2-part of the energy of the Coulomb system including the spin degree of freedom. The energy up to order e^2 of the two dimensional Coulomb system was calculated by Isihara and Toyoda [13]. It is calculated from the grand canonical potential up to order e^2 which is the summation of the grand canonical potential of an interaction free electron gas and the Coulomb exchange graph. Isihara and Toyoda calculated for this energy

$$\begin{aligned} U_{\mathrm{Coul}} = & \frac{\pi}{2} \frac{\rho^2}{m} + \frac{\pi}{12} m \frac{1}{\beta^2} - 0.067\, e^2 (2\pi\rho)^{\frac{3}{2}} \\ & + 0.063\, e^2 \frac{m^2}{(2\pi\rho)^{\frac{1}{2}}} \frac{1}{\beta^2} + 0.166\, e^2 \frac{m^2}{(2\pi\rho)^{\frac{1}{2}}} \frac{1}{\beta^2} \log\left(\frac{m}{\pi\rho\beta}\right). \end{aligned} \tag{42}$$

In Figure 2 we show the e^2-Coulomb energy of the $\nu = 1/2$ system in Hartree-Fock approximation as well as the e^2-Coulomb energy of the two dimensional Coulomb system (the spin polarized system as well as the system including the spin degree of freedom). It is seen from these curves that the energy curve of the CS gas has a much less steeper slope in comparison to the Coulomb gas including the spin degree of freedom. Furthermore, one sees from the figure that the main responsibility for this behaviour is the spin polarization of the CS gas. But also the μ and energy correction formulas of the CS gas of Section 2 increase this effect. The flatness of the energy curve can be understood by perturbational arguments. As mentioned below (27) $\langle H_0(B)\rangle$ does not contribute to the energy in e^2-order for $T \geq 0$. Thus the energy is given by the average value $\langle H_{ee}(B)\rangle$ calculated with respect to the Gibb's weight times the eigen functions of $H_0(B) + H_{ee}(B)$ which result from the lowest Landau level by Coulomb perturbations. It is easily seen that this term does not have any temperature corrections up to order e^2. Therefore, we get that the $\nu = 1/2$ Chern-Simons system does only pick up temperature corrections in higher $O(e^4)$ order in the parameter range (1). Summarizing, we get an agreement of the perturbational calculated energy (41) with this exact result.

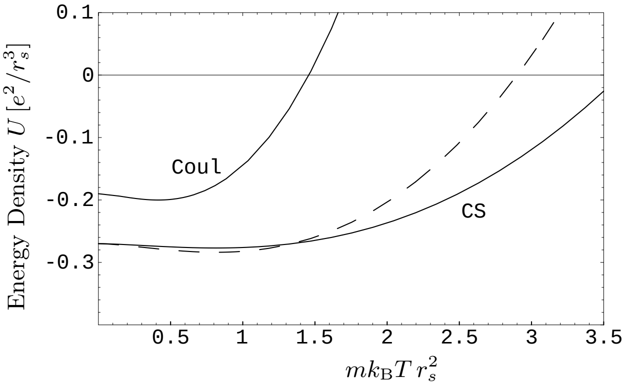

Fig. 2. The e^2 part of the energy density of the Coulomb- (Coul) as well as the $\nu = 1/2$ Chern-Simons (CS) gas. The energy density is given in units of e^2/r_s^3, where r_s is the electron radius $\rho = 1/(\pi r_s^2)$. The curve Coul is the e^2-energy density of the two dimensional Coulomb gas (42) including the spin degree of freedom. The curve CS is the e^2-energy density of the $\nu = 1/2$ Chern-Simons gas (41) originating from the spin polarized Coulomb exchange graph under consideration of the equations (35) and (37). The dashed curve is the e^2-energy density of a spin polarized two dimensional Coulomb gas.

3.2 The chemical potential and the energy of the $\nu = 1/2$ Chern-Simons system for $1/\beta \ll e^2\sqrt{\rho} \ll \rho/m$

As mentioned in the introduction of this paper the chemical potential and the energy in the parameter range (2) can not be calculated perturbationally through the Chern-Simons theory. HLR [1] and Kim, Lee [15] used the RPA for a resummation of diagrams to calculate the temperature corrections of the grand canonical potential in this parameter range. We will use in the following the result of their calculation to obtain the chemical potential and the energy of the $\nu = 1/2$ system. As in the last subsection we do not determine the first term in the expansion of the grand canonical potential in the parameter range (2) perturbationally. This term is given by the grand canonical potential of an electron gas in the lowest Landau level. This can be calculated through $\Omega_B(\mu\beta, (2\pi\tilde{\phi}\rho^*))$ limiting the summation in (31) to the lowest Landau level ($n = 0$). Since the higher Landau level terms contribute to the chemical potential and the energy with exponentially vanishing temperature corrections we will use in the following the full $\Omega_B(\mu\beta,(2\pi\tilde{\phi}\rho^*))$. Thus the grand canonical potential of the $\nu = 1/2$ system in the parameter range (2) is of the following form

$$\Omega(\mu,\beta) = \Omega_B(\mu,\beta,(2\pi\tilde{\phi}\rho^*)) + a'_1\, e^2 m^{\frac{3}{2}}\mu^{\frac{3}{2}} + a'_2\, m\mu\frac{1}{\beta} + a'_3\,\frac{1}{e^2} m^{\frac{1}{2}}\mu^{\frac{1}{2}}\frac{1}{\beta^2} + a'_4\,\frac{1}{e^2} m^{\frac{1}{2}}\mu^{\frac{1}{2}}\frac{1}{\beta^2}\log\left(\frac{1}{e^2\sqrt{m\mu}\beta}\right) + O(1/(e^4\beta^3)) + O(e^4 m^2\mu). \quad (43)$$

We now make the following ansatz for the terms of μ which scales polynomial with $1/\beta$:

$$\mu(\rho,\beta) = \pi\tilde{\phi}\frac{\rho}{m} + b'_1\, e^2\rho^{\frac{1}{2}} + b'_2\,\frac{1}{\beta} + b'_3\,\frac{1}{e^2}\frac{1}{\rho^{\frac{1}{2}}}\frac{1}{\beta^2} + b'_4\,\frac{1}{e^2}\frac{1}{\rho^{\frac{1}{2}}}\frac{1}{\beta^2}\log\left(\frac{1}{e^2\sqrt{\rho}\beta}\right). \quad (44)$$

Using the results of the last section (see the discussion below (28)) it is easy to see that the equations (34) and (36), determining the chemical potential and the energy are still valid for the approximation (43, 44). By using these equations we get with the help of the grand canonical potential (43) for the $\nu = 1/2$ system (as in the last subsection we set $b'_1 = 0$)

$$\mu = 2\pi\frac{\rho}{m}\cdot \quad (45)$$

The energy of the $\nu = 1/2$ system is given by

$$U = (2\pi)\frac{\rho^2}{m} + a'_1 e^2(2\pi\rho)^{\frac{3}{2}} - (a'_3 + a'_4)\,\frac{1}{e^2}(2\pi\rho)^{\frac{1}{2}}\frac{1}{\beta^2} - a'_4\frac{1}{e^2}(2\pi\rho)^{\frac{1}{2}}\frac{1}{\beta^2}\log\left(\frac{1}{e^2\sqrt{\rho}\beta}\right). \quad (46)$$

As mentioned above HLR [1] and Kim, Lee [15] calculated the temperature correction terms in (43) through a resummation of the RPA. The e^2-order term of the grand canonical potential in RPA for temperature $T = 0$ was calculated by us in [9]. By using these two results we get for the grand canonical potential (43)

$$\Omega_{\mathrm{RPA}}(\mu,\beta) = \Omega_B(\mu,\beta,(2\pi\tilde{\phi}\rho^*)) - 0.13\, e^2 m^{\frac{3}{2}}\mu^{\frac{3}{2}} - \frac{1}{6}\frac{1}{e^2}(2m\mu)^{\frac{1}{2}}\frac{1}{\beta^2}\log\left(\frac{1}{e^2\sqrt{m\mu}\beta}\right) + O\left(\frac{1}{e^2\sqrt{m\mu}\beta^2}\right). \quad (47)$$

By using this grand canonical potential we obtain from (45) and (46) up to the order $O((1/\beta^2)\log(1/e^2\sqrt{\rho}\beta))$

$$\mu^{\mathrm{RPA}} = 2\pi\frac{\rho}{m}, \quad (48)$$

$$U^{\mathrm{RPA}} = 2\pi\frac{\rho^2}{m} - 0.13\, e^2\,(2\pi\rho)^{\frac{3}{2}} + \frac{\sqrt{\pi}}{3}\frac{1}{e^2}\rho^{\frac{1}{2}}\frac{1}{\beta^2}\log\left(\frac{1}{e^2\sqrt{\rho}\beta}\right). \quad (49)$$

In [15] the temperature corrections to the ground state energy was calculated by $\partial(\beta\Omega_{\mathrm{RPA}})/(\partial\beta)\ (2\pi\rho/m,\beta)$. When comparing this expression with the temperature corrections of U^{RPA} (49) we get the same result. This is also correct for the full Ω (43). This is the reason for getting the same effective mass either by determining the effective mass through the one particle Green's function or through a comparison of the specific heat of the $\nu = 1/2$ system with the specific heat of an interaction free electron system (one can easily show from the equations above that this does not depend on the setting $b'_1 = 0$). The accordance of these two masses is well known in the case of the Coulomb system.

4 Conclusion

We considered in this paper at first the question whether corrections to the well known equations for determining in a Chern-Simons theory the chemical potential and the energy from a grand canonical potential in which one has eliminated the magnetic field and the field of the positive background through a mean field formalism should be taken into account. We showed that one gets corrections to these well known equations in contrast to the Coulomb system. We stated explicitly these corrections. Furthermore, we showed that these corrections can be determined from the grand canonical potential calculated by this mean field formalism. We should mention that our equations determining the chemical potential and the energy from this field eliminated grand canonical potential of the $\nu = 1/2$ system are also valid for other theories than the Chern-Simons theory of HLR.

With the help of these corrections and the well known result [13] for the grand canonical potential of the Coulomb exchange graph we calculated next the Hartree-Fock energy of the spin polarized $\nu = 1/2$ Chern-Simons system for low temperatures $T > 0$ up to order T^2. The parameter range of the calculated chemical potential and the energy is given by $e^2\sqrt{\rho} \ll 1/\beta \ll \rho/m$. We compared the energy with the energy of the two dimensional Coulomb gas taking into account the spin degrees of freedom. We get that the spin polarization as well as the corrections to the equations determining the chemical potential and the energy cause the energy as a function of the temperature to flatten. We showed that the exact behaviour of the energy within our approximation would have temperature corrections which are zero. Next we calculated the chemical potential and the energy from the temperature corrections of the grand canonical potential obtained in the RPA [1,15]. The parameter range of the chemical potential and the energy is given by $1/\beta \ll e^2\sqrt{\rho} \ll \rho/m$. We showed that one gets the same result for the energy as in the simplification [15] by using the chemical potential of the interaction free $\nu = 1/2$ system for temperature $T = 0$ and neglecting the corrections in the equation for determining the energy from the grand canonical potential.

Finally we mention that it should be in principle possible to calculate the energy of the $\nu = 1/2$ system for temperatures $T > 0$ through temperature heat capacity measurements. This was earlier done by Bayot *et al.* for the $\nu \approx 1$ system [20].

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